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Hasse diagrams and orbit class spaces

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ABSTRACT

Let X be a topological space and G be a group of homeomorphisms of X. Let $\tilde{G}$ be an equivalence relation on X defined by $x\tilde{G}y$ if the closure of the G-orbit of x is equal to the closure of the G-orbit of y. The quotient space $X/\tilde{G}$ is called the orbit class space and is endowed with the natural order inherited from the inclusion order of the closure of the classes, so that, if such a space is finite, one can associate with it a Hasse diagram. We show that the converse is also true: any finite Hasse diagram can be realized as the Hasse diagram of an orbit class space built from a dynamical system (X,G) where X is a compact space and G is a finitely generated group of homeomorphisms of X.

0. Introduction

Let X be a topological space and G be a group of homeomorphisms of X. The study of the properties of the G-orbits $\{g(x)\}_{g\in G}$, for all x in X, is in general a difficult problem also the orbit space X/G of the G-orbits is very complicated. For instance, if X is $\mathbb{S}^1$ and $G=\langle R_\alpha\rangle$ is the group generated by an irrational rotation R_α, then the orbit space can hardly be explicitly described. For this reason, instead of studying the orbit space X/G we can study a natural simpler one, called the orbit class space of (X,G) and denoted by $X/\tilde{G}$ which consists of all the orbit classes: two points of X belong to the same orbit class if the closures of their orbits are the same. In our previous example, since every G-orbit is dense in X, the orbit class space $X/\tilde{G}$ consists of only one element. Notice that the orbit space X/G of the above example is not T_0,[1] while the orbit class space $X/\tilde{G}$ is. This is true in general: the orbit class space $X/\tilde{G}$ is always T_0, but it needs not be T_1,[2] as shown by the following example. Let $Y=\{0,1\}\cup\{2^{-2^n},2^{-\frac{1}{2^n}}:\ n\in\mathbb{N}^*\}$ and G be the group generated by the homeomorphism of Y mapping any $y\in Y$ to y^2. In this case, the orbit space and the orbit class space coincide and they consist of 3 elements: the fixed orbit of 0, the fixed orbit of 1 and the orbit of $\frac{1}{2}$ whose closure is Y itself. The orbit class space is not T_1 because the orbit class of $\frac{1}{2}$ is not closed.

The space $X/\tilde{G}$ is simpler than X/G because $X/\tilde{G}$ is the T_0-ization of X/G (see [3]). Again the two spaces are very tightly linked together, since X/G can be mapped into $X/\tilde{G}$ by a quasi-homeomorphism [4]. Moreover, the structure of $X/\tilde{G}$ still

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[1] For any two points, there is an open set containing one of them but not the other.
[2] For any two points $[x]$ and $[y]$, there exist two open sets $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U}\ni[x]$, $\mathcal{U}\not\ni[y]$, $\mathcal{V}\ni[y]$ and $\mathcal{V}\not\ni[x]$.

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preserves information about the initial dynamical system (X, G) (see [2] for an application to the study of codimension-one foliations).

In order to describe some features of $X/\tilde{G}$, we associate with it an oriented graph Γ. Roughly speaking, the vertices will correspond to the orbit classes and there is an arc joining the vertex "class of x" to the vertex "class of y" if the closure of the orbit of x contains the closure of the orbit of y. In our last example (Y, G), the associated graph Γ consists of three vertices: $[0]$, $[1]$ and $[\frac{1}{2}]$, and two arcs: $[\frac{1}{2}] \to [0]$ and $[\frac{1}{2}] \to [1]$. In particular, if the orbit class space is finite, the associated graph is also finite and it can be described as the Hasse diagram related to a canonical order defined on $X/\tilde{G}$.

In this paper we take the opposite point of view: given a finite Hasse diagram Γ, do there always exist a compact space X and a finitely generated group G of homeomorphisms of X such that the graph associated with the orbit class space $X/\tilde{G}$ is the given Γ? We answer this question, asked in [1], in the affirmative.

Main Theorem. *Any finite connected Hasse diagram Γ can be realized as the orbit class space of a finitely generated group G of homeomorphisms of a compact connected topological space. Moreover, if the given Hasse diagram is a rooted tree of height n, then the underlying space X realizing Γ can be any compact connected manifold of dimension strictly greater than n.*

In particular, it follows from our construction that the realizing space X is a CW-complex. Furthermore, a minor modification of the construction guarantees that if the given Hasse diagram has a unique maximal element, then it can always be realized on a compact connected manifold. The question of whether any finite Hasse diagram can be realized on a compact manifold is still open.

Note that the constraints we focus on here concern finiteness: the Hasse diagram is finite, the underlying topological space is compact and the group of homeomorphisms is finitely generated. Note also that the finiteness of the original Hasse diagram is natural when dealing with compact spaces: the quotient of a compact space is quasi-compact.[3] Indeed the graph whose vertices are the integers and whose arcs join any integer to its predecessor, cannot be realized on a compact space.

Finally, the homeomorphisms we deal with here belong to a special class: they have "local" support, leave some subspaces invariant and are isotopic to the identity. If one is concerned with further conditions on the dynamics, let us mention that a more careful construction in the same spirit gives examples in which the appearing homeomorphisms satisfy stronger regularity conditions.

The paper is organized as follows. Section 1 contains background material: definitions and facts which we need throughout the paper. In Section 2 one can find the basic tools used in the proof of our Main Theorem. The fundamental notion of local realizability is discussed in Section 3. The realization result for trees on compact manifolds (Proposition 13) is proved by induction in Section 4. The construction giving the realization result for general graphs on compact spaces (Proposition 18) is described in Section 5.

1. Background

1.1. Orbit space and orbit class space

Let X be a topological space and G be a group of homeomorphisms of X. We define the *G-orbit of x* or, shortly, the *orbit of x*, as the set $G(x)$ of all the images of x under elements of G:

$$\text{orbit of } x = G(x) = \{g(x)\}_{g \in G}.$$

With this definition, one can consider the equivalence relation given by:

$$x \sim y \quad \text{if } G(x) = G(y).$$

The associated quotient space X/G is classically called the *orbit space* and it is naturally endowed with the quotient topology. The projection $\pi : X \to X/G$ defined by $\pi(x) = G(x)$ is open and onto.

According to [9], one can also define the *class of x* as:

$$\text{class of } x = [x] = \{y \in X \mid \overline{G(y)} = \overline{G(x)}\}.$$

The name "class" comes from the underlying equivalence relation:

$$x \approx y \quad \text{if } \overline{G(y)} = \overline{G(x)}.$$

The associated quotient space, denoted by $X/\tilde{G}$, is also endowed with the quotient topology. According to [2], we call $X/\tilde{G}$ the orbit class space.[4]

[3] A topological space X is called quasi-compact if for any open cover of X one can find a finite subcover. A quasi-compact space needs not be Hausdorff.

[4] In other contexts, this space is called the quasi-orbit space [8,1], but we choose to keep the initial definition in order to avoid ambiguities on the underlying dynamics [5].

The projection $\tilde{\pi} : X \to X/\tilde{G}$ defined by $\tilde{\pi}(x) = [x]$ is open and onto. The basic relation between this space and the previous one is that the map $\Pi : X/G \to X/\tilde{G}$ defined by $\Pi(G(x)) = [x]$ is onto, open and closed. Moreover, Π is a quasi-homeomorphisms, according to the Dieudonné–Grothendieck definition in [4].

From the topological point of view, we note that the space $X/\tilde{G}$ is Kolmogorov, that is, satisfies the T_0-separation axiom. As shown by the example in the introduction, in general, the space $X/\tilde{G}$ does not satisfy the T_1-separation axiom. We introduce the following order relation on $X/\tilde{G}$:

$$[x] \leqslant [y] \quad \text{if } \overline{G(y)} \subseteq \overline{G(x)}$$

which corresponds to the classical specialization order defined for any Kolmogorov space.

1.2. Hasse diagrams and orbit class spaces

Using the order relation defined above, we can represent the structure of $X/\tilde{G}$ in terms of oriented graphs. Since we want our graphs to be finite, we consider only finite orbit class spaces. Then the natural idea is to associate with the couple (X, G) an oriented graph Γ in the following way:

1. each class of $X/\tilde{G}$ corresponds to a vertex of Γ and vice versa;
2. there is an oriented edge from the vertex v associated with $[y]$ to the vertex w associated with $[x]$ if and only if $\overline{G(y)} \supsetneqq \overline{G(x)}$ and there is no class $[z]$ such that $\overline{G(x)} \subsetneqq \overline{G(z)} \subsetneqq \overline{G(y)}$.

In other terms, we associate with the ordered set $(X/\tilde{G}, \leqslant)$, assumed to be finite, its Hasse diagram, whose vertices are elements of the space $X/\tilde{G}$ itself, and the edges correspond to the covering relation. We can also remark that, necessarily, a graph Γ obtained this way from a couple (X, G) contains no oriented cycles. Moreover, if the underlying topological space X is connected, then the graph Γ must also be connected, disregarding the orientation. Since we can realize connected components individually, finite connected Hasse diagrams are the natural candidates (among finite oriented graphs) to represent the structure of the orbit class space $X/\tilde{G}$.

Definition 1. Let X be a topological space and G be a group of homeomorphisms of X. Let Γ be a finite connected Hasse diagram. We say that (X, G) realizes Γ if $(X/\tilde{G}, \leqslant)$ is finite and Γ is the Hasse diagram of $(X/\tilde{G}, \leqslant)$.

Among all graphs, trees play a special role (we recall that a non-oriented graph is called a *tree* if it is connected and acyclic, and that a disjoint union of trees is called a *forest*). We say that a (non-oriented) tree is *rooted* if it has a distinguished vertex, called the *root*. The choice of a root v endows the tree with a natural orientation: v is the one vertex such that the unique chain connecting v to an other vertex w is an oriented path from v to w. Next, we shall shortly say that a given oriented graph Γ is a *rooted tree* if, disregarding the orientation, it is a rooted tree and if the original orientation of the edges of Γ coincides with the orientation induced by the choice of the root, as described above. Notice that a rooted tree is automatically a (finite and connected) Hasse diagram.

It will be useful in our proofs to organize the vertices of a Hasse diagram according to some hierarchical criterion. Let Γ be a Hasse diagram, we define its *floors* by:

$$F0 =: \textit{ground floor} = \{\text{vertices with no outgoing edges}\};$$

$$F1 =: \textit{first floor} = \{\text{vertices whose direct successors belong to } F0\};$$

$$Fn =: \textit{n-th floor} = \{\text{vertices whose direct successors lie on the previous } n \text{ floors and at least one of such successors belongs to } F(n-1) \text{ for } n \geqslant 1\}.$$

Recall that the vertex w is called a *direct successor* of a vertex v if there is an oriented edge from v to w. The *height of a vertex* is a positive integer n if it lies in the n-th floor, while the *height of a graph* is the maximum of the heights of all its vertices. If the vertex whose height is equal to the height of the graph is unique, then we say that the graph has a *unique maximal element*.

1.3. A standard construction

For any topological space X, let $\mathbf{1}_X$ (or shortly, $\mathbf{1}$) be the identity homeomorphism defined on X. Let g be a map defined on X and Y be a subset of X, we denote by $g|_Y$ the restriction of g to Y. We recall that an *isotopy* of a homeomorphism f of a topological space X is a continuous map $\Phi : X \times I \to X$ such that:

- for all $x \in X$, $\Phi(x, 0) = f(x)$;
- for all $t \in I$, the map $\Phi_t(\cdot) = \Phi(\cdot, t)$ is a homeomorphism of X.

Moreover, if $\Phi(x,1)=x$ for all $x\in X$, then we say that f is *isotopic to the identity* $\mathbf{1}_X$ on X. For practical reasons we introduce the following definition.

Definition 2. Let X be a topological manifold with non-empty boundary ∂X, and f be a homeomorphism of X such that $f|_{\partial X}=\mathbf{1}_{\partial X}$ and which is isotopic to the identity via the isotopy Φ. We call such an isotopy $\mathbf{1}_\partial$-preserving if for all $x\in\partial X$ and for all $t\in I$, the map $\Phi_t(\cdot)=\Phi(\cdot,t)$ coincides with the identity $\mathbf{1}_{\partial X}$. In this case, the homeomorphism f is said to be $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_X$.

Next we will use the following practical two-step elementary construction.

Lemma 3. *Let f be a homeomorphism of a topological space X which is isotopic to the identity $\mathbf{1}_X$ via the isotopy Φ. Let $\tilde{\Phi}$ be the isotopy from the identity $\mathbf{1}_X$ to itself defined by*:

$$\begin{aligned}\tilde{\Phi}&:X\times I\to X\\ &(x,t)\to\tilde{\Phi}(x,t)=\begin{cases}\Phi(x,1-3t) & \text{if } t<\frac{1}{3};\\ f(x) & \text{if } \frac{1}{3}\leqslant t\leqslant\frac{2}{3};\\ \Phi(x,3t-2) & \text{if } t>\frac{2}{3}.\end{cases}\end{aligned}$$

Then the map $\hat{f}$ defined from the isotopy $\tilde{\Phi}$ by:

$$\begin{aligned}\hat{f}&:X\times I\to X\times I\\ &(x,t)\to\big(\tilde{\Phi}(x,t),t\big)\end{aligned}$$

is a homeomorphism of $X\times I$ satisfying:

(1) *the restriction of $\hat{f}$ to $X\times\{0,1\}$ coincides with the identity $\mathbf{1}_{X\times\{0,1\}}$ and $\hat{f}$ is isotopic to the identity $\mathbf{1}_{X\times I}$*;
(2) *if $\partial X\neq\emptyset$ and Φ is $\mathbf{1}_\partial$-preserving, then the restriction of $\hat{f}$ to the whole boundary of $X\times I$ coincides with the identity $\mathbf{1}_{\partial(X\times I)}$ and $\hat{f}$ is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{X\times I}$.*

Proof. The proof is straightforward. Notice that $\tilde{\Phi}$ is symmetric in t, that is, $\tilde{\Phi}(x,t)=\tilde{\Phi}(x,1-t)$. The existence of an isotopy towards $\mathbf{1}_{X\times I}$ with the announced properties can be obtained by considering:

$$\begin{aligned}\hat{\Phi}&:(X\times I)\times I\to X\times I\\ &\big((x,t),s\big)\to\hat{\Phi}\big((x,t),s\big)=\begin{cases}(\tilde{\Phi}(x,t-s),t) & \text{if } 0\leqslant s\leqslant t \text{ and } t\leqslant\frac{1}{2};\\ (\tilde{\Phi}(x,t+s),t) & \text{if } 0\leqslant s\leqslant 1-t \text{ and } t>\frac{1}{2};\\ (x,t) & \text{otherwise.}\end{cases}\qquad\square\end{aligned}$$

2. Preliminary lemmas

In this section we construct some useful finitely generated groups of homeomorphisms of some standard topological spaces.

For any space X, let $\mathrm{Homeo}(X)$ be the set of all the homeomorphisms of X. For the reader's convenience, we add to each lemma a sentence roughly related to the content of the lemma itself.

The density of diffeomorphisms of the following lemma is well known (see [7, p. 305] and [6]).

Lemma 4 *(Density on the interval). Let I be the closed interval* $[0,1]$. *There exists a finitely generated Abelian group G of homeomorphisms of I such that*:

(1) *for all $g\in G$, we have $g|_{\partial I}=\mathbf{1}_{\{0,1\}}$, that is, $g(0)=0$ and $g(1)=1$*;
(2) *for all $x\in Int(I)$, we have $\overline{G(x)}=I$ and $[x]=Int(I)$*;
(3) *for all $g\in G$, g is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_I$.*

Proof. Let T_1 be the translation defined on $\mathbf{R}$ by $T_1(x)=x+1$ and $T_{\sqrt{2}}$ be the translation defined on $\mathbf{R}$ by $T_{\sqrt{2}}(x)=x+\sqrt{2}$. Let $h:Int(I)\to\mathbf{R}$ be an increasing homeomorphism. Then $\varphi=h^{-1}\circ T_1\circ h$ and $\psi=h^{-1}\circ T_{\sqrt{2}}\circ h$ are increasing homeomorphisms of $Int(I)$ which can be extended to I by $\varphi(0)=\psi(0)=0$ and $\varphi(1)=\psi(1)=1$. The group $G\subset\mathrm{Homeo}(I)$ generated by the homeomorphisms φ and ψ defined above, that is,

$$G=\langle\varphi,\psi\rangle$$

is Abelian by construction (translations of $\mathbf{R}$ commute) and the orbit of any $x \in Int(I)$ is dense in I since the corresponding translation values 1 and $\sqrt{2}$ are rationally independent. To show Item 3, just consider the isotopy $F : I \times I \to I$ defined by $F(x,t) = tx + (1-t)g(x)$ where g belongs to the set of generators $\{\varphi, \psi\}$. $\square$

Lemma 5 *(Density of rotations on the n-dimensional sphere). For all integers $n \geqslant 1$, let $\mathbf{S}^n$ be the n-dimensional sphere. There exists a finitely generated group G of homeomorphisms of $\mathbf{S}^n$ such that*:

(1) *for all $x \in \mathbf{S}^n$, we have $\overline{G(x)} = \mathbf{S}^n$ and $[x] = \mathbf{S}^n$;*
(2) *for all $g \in G$, g is isotopic to the identity $\mathbf{1}_{\mathbf{S}^n}$.*

Proof. Let $\mathbf{S}^n = \{(x_1,\dots,x_{n+1}) \in \mathbf{R}^{n+1} \mid \sum_{j=1}^{n+1} x_j^2 = 1\}$ be the n-dimensional sphere and α be an element of $\mathbf{R}$ rationally independent of 2π. For all $i \in \{1,\dots,n\}$, let $R_{\alpha,i}$ be the rotation of angle α on the 1-sphere of $\mathbf{S}^n$ parallel to the $(i,i+1)$-plane and defined by the square matrix of size $n+1$, whose only non-zero coefficients are:

- $a_{j,j} = 1$ for all $j \in \{1,\dots,n+1\}$, $j \neq i$ and $j \neq i+1$;
- $\begin{bmatrix} a_{i,i} & a_{i,i+1} \\ a_{i+1,i} & a_{i+1,i+1} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Let G be the group of homeomorphisms of $\mathbf{S}^n$ generated by all such rotations:

$$G = \langle R_{\alpha,i},\ i = 1,\dots,n\rangle.$$

We proceed inductively on n (n is the dimension of the sphere).

For $n=1$, G is generated by a rotation R_α of angle α. Since α is rationally independent of 2π, for all $(x_1,x_2) \in \mathbf{S}^1$ we have $\overline{\{R_\alpha^k((x_1,x_2))\}_{k\in\mathbf{Z}}} = \mathbf{S}^1$. Moreover, R_α is isotopic to the identity $\mathbf{1}_{\mathbf{S}^1}$ via the isotopy $F : \mathbf{S}^1 \times I \to \mathbf{S}^1$ defined by:

$$F\big((x_1,x_2),t\big) = R_{(1-t)\alpha}\big((x_1,x_2)\big)$$

where $R_{(1-t)\alpha}$ is the rotation of angle $(1-t)\alpha$ on $\mathbf{S}^1$.

Let us now assume, by induction, that our statement is true for all $k \in \{1,\dots,n-1\}$ and let us show it is true for $k=n$.

Item 1: for any $\underline{y} = (y_1,\dots,y_{n+1}) \in \mathbf{S}^n$ and $\underline{z} = (z_1,\dots,z_{n+1}) \in \mathbf{S}^n$ we shall show that there is an iterate of $\underline{y}$ under G arbitrarily close to $\underline{z}$. First, by applying the hypothesis of induction for $k=1$, up to a rotation $R_{\alpha,1}^l$ ($l \in \mathbf{Z}$) of the 1-sphere $\{\underline{x} \in \mathbf{R}^{n+1} \mid x_1^2 + x_2^2 = 1 - \sum_{j=3}^{n+1} y_j^2\}$, we can assume that:

$$\underline{y} = \left(\varepsilon, \sqrt{1 - \varepsilon^2 - \sum_{j=3}^{n+1} y_j^2}, y_3, \dots, y_{n+1}\right)$$

for ε arbitrarily small. In the same way, we can assume that:

$$\underline{z} = \left(\tilde{\varepsilon}, \sqrt{1 - \tilde{\varepsilon}^2 - \sum_{j=3}^{n+1} z_j^2}, z_3, \dots, z_{n+1}\right)$$

for $\tilde{\varepsilon}$ small enough and arbitrarily close to ε. Second, Let $\underline{w} = (\varepsilon, w_2, \dots, w_{n+1})$ be close enough to $\underline{z}$. By using induction on the $(n-1)$-sphere $\{\underline{x} = (\varepsilon, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \sum_{j=2}^{n+1} x_j^2 = 1 - \varepsilon^2\}$, there exists an element g of the group generated by $\{R_{\alpha,i}\}_{i=2}^n$ such that $g(\underline{y})$ is arbitrarily close to $\underline{w}$ and we are done.

Item 2: All the generators are isotopic to $\mathbf{1}_{\mathbf{S}^n}$ via the isotopy $F : \mathbf{S}^n \times I \to \mathbf{S}^n$ defined by:

$$F(\underline{x}, t) = R_{(1-t)\alpha,i}(\underline{x}). \qquad \square$$

Lemma 6 *(Density on the collars of n-dimensional spheres). For all integers $n \geqslant 1$, let X_n be the collar of an n-dimensional sphere defined by $X_n = \mathbf{S}^n \times I$. Then there exists a finitely generated group G of homeomorphisms of X_n such that*:

(1) *for all $g \in G$, we have $g|_{\partial X_n} = \mathbf{1}_{\partial X_n}$ where $\partial X_n = \mathbf{S}^n \times \{0,1\}$;*
(2) *for all $x \in Int(X_n)$, we have $\overline{G(x)} = X_n$ and $[x] = Int(X_n)$;*
(3) *for all $g \in G$, g is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{X_n}$.*

Proof. Let G_5 be the group of homeomorphisms of $\mathbf{S}^n$ as defined in Lemma 5. Apply to every generator g of G_5 the two-step construction as defined in Lemma 3 in order to create the associated homeomorphism $\hat{g}$ of $\mathbf{S}^n \times I$. Let $\hat{S}_5$ be the set of generators obtained in this way:

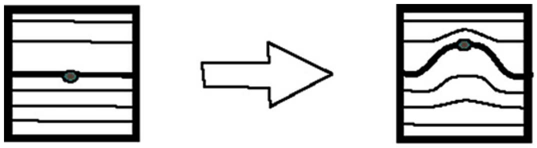

Fig. 1. The qualitative behavior of ρ (Lemma 7) and θ (Lemma 11).

$$\hat{S}_5 = \{\hat{g} \mid g \text{ generator of } G_5\}.$$

Note that, using Lemma 3, the elements of $\hat{S}_5$ satisfy Items 1 and 3.

Now, let G_4 be the group of homeomorphisms of I generated by the two homeomorphisms φ and ψ as defined in Lemma 4. For all generators g of G_4, let $\mathbf{1}_{\mathbf{S}^n} \times g$ be the homeomorphism of $\mathbf{S}^n \times I$ defined by:

$$(\mathbf{1}_{\mathbf{S}^n} \times g)(\underline{x}, t) = \big(\underline{x}, g(t)\big)$$

and S'_4 be the set of the two homeomorphisms obtained in this way:

$$S'_4 = \{\mathbf{1}_{\mathbf{S}^n} \times \varphi, \mathbf{1}_{\mathbf{S}^n} \times \psi\}.$$

The elements of S'_4 satisfy Item 1 by construction and Item 3 by considering the isotopy $1_{\mathbf{S}^n} \times F$ where F isotopes φ (resp. ψ) to $\mathbf{1}_I$.

Finally, let G be the group generated by the elements of $\hat{S}_5$ and S'_4:

$$G = \big\langle \hat{S}_5, S'_4 \big\rangle.$$

The group G is finitely generated by construction and its elements satisfy Items 1 and 3 because the generators do. To show that Item 2 is also satisfied, let $(\underline{x}, t_x)$ and $(\underline{z}, t_z)$ be two arbitrary points of $\mathbf{S}^n \times I$. Starting from $(\underline{x}, t_x)$, by an element of $\langle S_4 \rangle$ one can obtain a point of coordinates $(\underline{x}, t_{middle})$ with $t_{middle} \in [\frac{1}{3}, \frac{2}{3}]$; from $(\underline{x}, t_{middle})$, by an element of $\langle \hat{S}_5 \rangle$ one can obtain $(\underline{y}, t_{middle})$ with $\underline{y}$ is arbitrarily close to $\underline{z}$; finally, from $(\underline{y}, t_{middle})$, by another element of $\langle S_4 \rangle$ one can obtain a point $(\underline{y}, t_y)$ with t_y is arbitrarily close to t_z and we are done. $\square$

Lemma 7 *(Density on all n-dimensional balls). For all integers $n \geqslant 1$, let $\mathbf{D}^n$ be the n-dimensional closed ball and $\partial \mathbf{D}^n$ be its boundary $\mathbf{S}^{n-1}$. Then there exists a finitely generated group G of homeomorphisms of $\mathbf{D}^n$ such that*:

(1) *for all $g \in G$, we have $g|_{\partial \mathbf{D}^n} = g|_{\mathbf{S}^{n-1}} = \mathbf{1}_{\mathbf{S}^{n-1}}$;*
(2) *for all $x \in Int(\mathbf{D}^n)$, we have $\overline{G(x)} = \mathbf{D}^n$ and $[x] = Int(\mathbf{D}^n)$;*
(3) *for all $g \in G$, g is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{\mathbf{D}^n}$.*

Proof. For $n = 1$, this lemma is Lemma 4.

For $n > 1$, let us identify $\mathbf{D}^n$ with the quotient space $(\mathbf{S}^{n-1} \times I)/_{(\mathbf{S}^{n-1} \times \{0\}) = \{p\}}$. Note that the homeomorphisms of $\hat{S}_5$ and S'_4 (defined in the proof of Lemma 6) pass to the quotient and the quotiented homeomorphisms satisfy Items 1 and 3 of Lemma 7. Let us denote by SQ the set of the homeomorphisms obtained in this way:

$$SQ = \{g/_{(\mathbf{S}^{n-1} \times \{0\}) = \{p\}} \mid g \in S_5 \cup S_4\}.$$

Note that p is a fixed point for each element of S. Let ρ be a homeomorphism of $\mathbf{D}^n$ such that:

- $\rho(p) \neq p$;
- $\rho|_{\partial \mathbf{D}^n} = \mathbf{1}_{\mathbf{S}^{n-1}}$;
- ρ is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{\mathbf{D}^n}$

(see Fig. 1). For instance, if we look at $\mathbf{D}^n$ as $\mathbf{D}^{n-1} \times I$ where $\mathbf{D}^{n-1}$ is the $(n-1)$-dimensional disk of radius 1, one can take:

$$\rho : \mathbf{D}^{n-1} \times I \to \mathbf{D}^{n-1} \times I$$
$$(x, u) \to \begin{cases} (x, ue^{u(\|x\|^2 - 1)}) & \text{if } 0 \leqslant u \leqslant \frac{1}{2}; \\ (x, ue^{(1-u)(\|x\|^2 - 1)}) & \text{if } \frac{1}{2} \leqslant u \leqslant 1. \end{cases}$$

The group G defined by:

$$G = \langle SQ, \rho \rangle$$

is finitely generated by construction and its elements satisfy Items 1 and 3 because the generators do. Let $x \in Int(\mathbf{D}^n)$, up to applying ρ, we can assume that $x \neq p$, so that the density of its orbit follows from Item 2 of Lemma 6 which implies Item 2 of this lemma. $\square$

Lemma 8 *(Density on all n-dimensional balls with holes). For all integers $n \geqslant 2$ and $k \geqslant 1$, let*

$$Y_{n,k} = \mathbf{S}^n \Big\backslash \left(\bigcup_{i=1}^{k} Int(A_i) \right)$$

where all A_i are pairwise disjoint and homeomorphic to $\mathbf{D}^n$. Then there exists a finitely generated group G of homeomorphisms of $Y_{n,k}$ such that:

(1) *for all $g \in G$, we have $g|_{\partial Y_{n,k}} = \mathbf{1}_{\partial Y_{n,k}}$ where $\partial Y_{n,k} = \bigcup_{i=1}^{k} \partial A_i$;*
(2) *for all $x \in Int(Y_{n,k})$, we have $\overline{G(x)} = Y_{n,k}$ and $[x] = Int(Y_{n,k})$;*
(3) *for all $g \in G$, g is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{Y_{n,k}}$.*

Proof. For all $i = 1, \dots, k$, let $C_i \subseteq Y_{n,k}$ be a collar of ∂A_i, that is, C_i is homeomorphic to $\mathbf{S}^{n-1} \times I$ and ∂A_i is identified with $\mathbf{S}^{n-1} \times \{0\}$. Assume that all C_i are pairwise disjoint and denote by C their disjoint union $C = \bigcup_{i=1}^{k} C_i$. Let S_C be the set:

$$S_C = \left\{ s \in \text{Homeo}(Y_{n,k}) \text{ such that } s|_{Y_{n,k} \setminus C} = \mathbf{1}_{Y_{n,k} \setminus C} \text{ and } s|_{C_i} = g,\ i = 1, \dots, k \right\}_{g \in S_6}$$

where S_6 is the finite set of generators of the group built in Lemma 6. Note that for all i, the orbit under the group $\langle S_C \rangle$ of a point $x \in Int(C_i)$ is dense in C_i.

In order to make these orbits dense in the whole space, we shall enrich the set of the generators in the following way. Let $\{\mathcal{V}_\beta\}_{\beta \in B}$ be an open cover of the closure of $(Y_{n,k} \setminus C)$ such that: for all $\beta \in B$, $\mathcal{V}_\beta \subseteq Y_{n,k}$ is homeomorphic to $Int(\mathbf{D}^n)$ and is disjoint from $\partial Y_{n,k}$. By compactness of $Y_{n,k} \setminus C$, we can assume that the index set B is finite. For all $\beta \in B$, let S_β be the set:

$$S_\beta = \left\{ s \in \text{Homeo}(Y_{n,k}) \text{ such that } s|_{Y_{n,k} \setminus \mathcal{V}_\beta} = \mathbf{1}_{Y_{n,k} \setminus \mathcal{V}_\beta} \text{ and } s|_{\mathcal{V}_\beta} = g \right\}_{g \in S_7}$$

where S_7 is the finite set of generators of the group built in Lemma 7. The group G defined by:

$$G = \left\langle S_C, (S_\beta, \beta \in B) \right\rangle$$

is therefore finitely generated and its elements satisfy Items 1 and 3 because the generators do. To show that Item 2 holds, just consider that by connectedness, for any two points x and y in $Int(Y_{n,k})$, there exists a finite chain $\{\mathcal{O}_j\}_{j=0}^{m}$ of open balls either contained in one of the collars C_i or in one of the disks $\mathcal{V}_\beta$ such that $\mathcal{O}_0$ contains x, $\mathcal{O}_m$ contains y and for all $j = 1, \dots, m$ the intersection $\mathcal{O}_{j-1} \cap \mathcal{O}_j$ is non-empty. Since by construction the G-orbit of any point contained in any $\mathcal{O}_j$ is dense in $\mathcal{O}_j$, we are done. $\square$

3. Local and global realizability of graphs

In this section we introduce the notion of *local realizability* of a graph which plays a central role for us. When we realize a graph we do it locally in order to avoid constraints on the underlying topological space X and the group of actions G. Lemma 10 and Corollary 12 show that this is a good strategy.

Definition 9 *(Local realizability of Γ in dimension n).* Let Γ be an oriented graph and n be an integer greater or equal to 1. We say that Γ is locally realizable in dimension n if there exists a finitely generated group $G_{loc,n}$ of homeomorphisms of $\mathbf{D}^n$ with the following properties:

(1) $(Int(\mathbf{D}^n), G_{loc,n})$ realizes Γ;
(2) for all $g \in G_{loc,n}$ we have $g|_{\partial \mathbf{D}^n} = \mathbf{1}_{\mathbf{S}^{n-1}}$;
(3) for all $g \in G_{loc,n}$, g is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{\mathbf{D}^n}$;
(4) there exists an open ball $B_{loc,n}$ relatively compact in $Int(\mathbf{D}^n)$ such that each vertex v of Γ has a representative in $B_{loc,n}$ (that is, v corresponds to a class $[x]$ such that $x \in B_{loc,n}$);
(5) for all x in $Int(\mathbf{D}^n) \setminus B_{loc,n}$, there exists a unique vertex v_0 of Γ such that $[x] = v_0$.

Let Γ be a graph having a unique maximal vertex m. If Γ is locally realizable, then the vertex v_0 appearing in the definition must coincide with m. Indeed, by definition the class associated with m is the whole space $\mathbf{D}^n$.

The following lemmas explain the generality which we gain when we take the local point of view.

Lemma 10. *If the graph Γ is locally realizable in dimension n, then for all n-dimensional compact connected manifolds M there exists a finitely generated group G of homeomorphisms of M isotopic to the identity $\mathbf{1}_M$ such that (M, G) realizes Γ.*

Proof. The proof is based on a compactness argument. More precisely, let $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ be an open cover of M such that for all $\alpha \in A$, $\mathcal{U}_\alpha$ is homeomorphic to $Int(\mathbf{D}^n)$. By compactness of M, we can assume that the index set A is finite. There exists $\alpha_0 \in A$ such that for all $\alpha \in A$, $\alpha \neq \alpha_0$, $\mathcal{U}_\alpha$ does not intersect an open ball B relatively compact in $\mathcal{U}_{\alpha_0}$. Thus, there exists a positive integer k such that $\{\mathcal{U}_{\alpha_i}\}_{i=1}^k$ is a finite cover of $M \setminus B$. We have that $\{\mathcal{U}_{\alpha_i}\}_{i=0}^k$ is a finite cover of M. For $\alpha = \alpha_0$, let S_{α_0} be the set:

$$S_{\alpha_0} = \left\{s \in \mathrm{Homeo}(M) \text{ such that } s|_{M\setminus\mathcal{U}_\alpha} = \mathbf{1}_{M\setminus\mathcal{U}_\alpha} \text{ and } s|_{\mathcal{U}_\alpha} = g\right\}_{g\in S_{loc}}$$

where S_{loc} is the finite set of generators of $G_{loc,n}$ (Definition 9). For all $i \in \{1,\dots,k\}$, let S_{α_i} be the set:

$$S_{\alpha_i} = \left\{s \in \mathrm{Homeo}(M) \text{ such that } s|_{M\setminus\mathcal{U}_\alpha} = \mathbf{1}_{M\setminus\mathcal{U}_\alpha} \text{ and } s|_{\mathcal{U}_\alpha} = g\right\}_{g\in S_7}$$

where S_7 is the finite set of generators of the group built in Lemma 7. The group G defined by:

$$G = \langle S_{\alpha_i},\ i = 0,\dots,k\rangle$$

is therefore finitely generated and all its generators are isotopic to the identity $\mathbf{1}_M$. Note that, by connectedness of $M \setminus \mathcal{U}_{\alpha_0}$, for all x in $M \setminus \mathcal{U}_{\alpha_0}$ we have $[x] = v_0$ where v_0 is the vertex appearing in Definition 9, so that the orbit class graph associated with (M,G) is exactly Γ. □

Lemma 11. *Let Γ be a graph with a unique maximal element and assume that Γ is locally realizable in dimension n. Then for all integers $m \geqslant n$, the graph Γ is also locally realizable in dimension m.*

Proof. It suffices to prove that if Γ is locally realizable in dimension k, then Γ is locally realizable in dimension $k+1$. For this purpose, we imbed $\mathbf{D}^k$ in $\mathbf{D}^{k+1}$ such that the $\mathbf{D}^k$ on which we have the local realization in dimension k will always be identified with $\mathbf{D}^k \times \{\frac{1}{2}\} \subset \mathbf{D}^{k+1}$.

Since $\mathbf{D}^{k+1}$ is homeomorphic to $\mathbf{D}^k \times I$, following the construction of Lemma 3 we can associate with any homeomorphism g of $G_{loc,k}$ a homeomorphism $\hat{g}$ of $\mathbf{D}^{k+1}$ satisfying Items 1 and 2 of Definition 9. In particular, let us define:

$$\hat{S} = \{\hat{g} \mid g \text{ generator of } G_{loc,k}\}.$$

Recall that for all $t \in I$, $\hat{g}$ leaves invariant each fiber $\mathbf{D}^k \times \{t\}$, and so for $t = \frac{1}{2}$, the restriction of $\hat{g}$ to $\mathbf{D}^k \times \{\frac{1}{2}\}$ coincides with g.

We can also look at $\mathbf{D}^{k+1}$ as $(\mathbf{D}^k \times [0,\frac{1}{2}]) \bigcup_{\mathbf{D}^k\times\{\frac{1}{2}\}} (\mathbf{D}^k \times [\frac{1}{2},1])$, each of the two appearing spaces is homeomorphic to a $(k+1)$-dimensional disk. Consider the set:

$$S = \left\{s \in \mathrm{Homeo}\big(\mathbf{D}^{k+1}\big) \text{ such that } s|_{\mathbf{D}^k\times[0,\frac{1}{2}]} = s|_{\mathbf{D}^k\times[\frac{1}{2},1]} = g\right\}_{g\in S_7}$$

where S_7 is the finite set of generators of the group built in Lemma 7. Note that the restriction of all $s \in S$ to $\mathbf{D}^k \times \{\frac{1}{2}\}$ coincides with the identity map $\mathbf{1}_{\mathbf{D}^k}$.

Finally, let O_k be homeomorphic to a k-dimensional disk and contained in $Int(\mathbf{D}^k \setminus B_{loc,k}) \times \{\frac{1}{2}\}$ where $B_{loc,k}$ is as in Definition 9. Denote by O_{k+1} the set $O_k \times [\frac{1}{3},\frac{2}{3}]$ and consider a homeomorphism θ of $\mathbf{D}^{k+1}$ such that:

- $\theta|_{\mathbf{D}^{k+1}\setminus Int(O_{k+1})} = \mathbf{1}_{\mathbf{D}^{k+1}\setminus Int(O_{k+1})}$;
- $\theta(Int(O_k)) \cap O_k = \emptyset$;
- θ is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{\mathbf{D}^{k+1}}$, the isotopy having support in O_{k+1}

(see Fig. 1). The role of θ is to map points of $\mathbf{D}^k \times [0,\frac{1}{2}]$ to $\mathbf{D}^k \times [\frac{1}{2},1]$ and vice versa, by θ^{-1}, as well as to map all the points of O_k to one of the two spaces above.

The group G defined by:

$$G_{loc,k} = \langle \hat{S}, S, \theta\rangle$$

is therefore finitely generated and satisfies Items 2 and 3 of Definition 9 because all the generators do. Item 5 holds: on one hand the orbits of all the points x of $\mathbf{D}^{k+1} \setminus (\mathbf{D}^k \times \{\frac{1}{2}\})$ are dense in $\mathbf{D}^{k+1}$, on the other hand, if y is a point of $(\mathbf{D}^k \setminus B_{loc,k}) \times \{\frac{1}{2}\}$, by applying θ we can insure that no new class has been created. This argument also shows Items 1 and 4. □

By applying Lemma 10 to Lemma 11, we directly get the following corollary.

Corollary 12. *Let Γ be a graph with a unique maximal element and assume that Γ is locally realizable in dimension n. Then for all integers $m \geqslant n$ and all compact connected m-dimensional manifolds, there exists a finitely generated group G of homeomorphisms of M isotopic to the identity $\mathbf{1}_M$ such that (M,G) realizes Γ.*

4. Realization of rooted trees by compact manifolds

The aim of this section is to prove the following result:

Proposition 13. *Let Γ be a rooted tree of height n and m be an integer strictly greater than n. Then, for any m-dimensional compact connected manifold M, there exists a finitely generated group $G \subseteq \mathrm{Homeo}(M)$ such that (M, G) realizes Γ.*

The proof of Proposition 13 follows by induction on the height of the tree. The initialization will be discussed in Section 4.1, while the induction steps will be described in Section 4.2.

4.1. Trees of small height

The following facts, concerning trees of height 0 and 1, are immediate consequences of the constructions of the previous sections. The next lemma asserts that any n-dimensional manifold has a minimal group.

Lemma 14. *Let $\Gamma = \{v\}$ be a graph containing a single vertex and no edges. Then for all integers $n \geqslant 1$,*

(1) *Γ is locally realizable in dimension n;*
(2) *Γ is realizable on any compact n-dimensional manifold.*

Proof. The proof is straightforward. Item 1 follows either directly from Lemma 7 or by induction using Lemma 4 together with Lemma 11. Item 2 follows from Corollary 12. □

The forthcoming lemma shows how to realize trees of height 1 on manifolds of dimension at least 2.

Lemma 15. *Let Γ be a rooted tree of height 1. Then for all integers $n \geqslant 2$,*

(1) *Γ is locally realizable in dimension n;*
(2) *Γ is realizable on any compact n-dimensional manifold.*

Proof. Let v_0 be the root of a tree Γ of height 1 and $(w_1, \ldots, w_{k(\Gamma)})$ be the vertices of the first floor of Γ. For $n \geqslant 2$, apply Lemma 8 to $Y_{n,k}$ for $k = k(\Gamma) + 1$. Denote by G_8 the associated group. The quotient space $Y_{n,k}/_{\partial A_i = \{x_i\},\ i=1,\ldots,k(\Gamma)}$ obtained by identifying each ∂A_i with a point denoted by x_i, is then homeomorphic to $\mathbf{D}^n$. Let $G_{loc,n}$ be defined by considering the homeomorphisms of G_8 after making the quotient on $Y_{n,k}$, that is,

$$G_{loc,n} = \left\{h \in \mathrm{Homeo}\big(\mathbf{D}^n\big) \text{ such that } h|_{\partial \mathbf{D}^n \cup \{x_1,\ldots,x_{k(\Gamma)}\}} = \mathbf{1}_{\partial \mathbf{D}^n \cup \{x_1,\ldots,x_{k(\Gamma)}\}} \text{ and } h|_{Int(\mathbf{D}^n)\setminus\{x_1,\ldots,x_{k(\Gamma)}\}} = g|_{Int(Y_{n,k})}\right\}_{g \in G_8}.$$

Then, by construction, for all $i = 1, \ldots, k(\Gamma)$ we have $[x_i] = w_i$, while for all $x \in Int(\mathbf{D}^n)$, $x \neq x_i$, we have $[x] = v_0$. □

4.2. Induction: trees of any height

Before describing the induction for the local realizability of a tree of any height (Lemma 17), we prove a technical lemma.

Lemma 16. *For all integers $n \geqslant 2$ and $k \geqslant 1$, let $X_{n+1,k}$ be a topological space homeomorphic to $\mathbf{D}^{n+1} = (\mathbf{S}^n \times I)/_{\mathbf{S}^n \times \{0\} = \{p\}}$ and provided with a marked subspace $Y_{n,k}$ defined as follows. Let A_i, $i = 1, \ldots, k$, be k pairwise disjoint sets which are homeomorphic to $\mathbf{D}^n$ and lie in $\mathbf{S}^n \times \{\frac{1}{2}\} \subseteq X_{n+1,k}$. Let us denote by $Y_{n,k}$ the set $(\mathbf{S}^n \times \{\frac{1}{2}\}) \setminus (\bigcup_{i=1}^k Int(A_i))$. Then there exists a finitely generated group G of homeomorphisms of $X_{n+1,k}$ such that:*

(1) *for all $g \in G$, we have $g|_{\partial X_{n+1,k}} = \mathbf{1}_{\mathbf{S}^n}$;*
(2) *the behavior of the G-orbits is the following:*
- *for all $x \in Int(X_{n+1,k} \setminus Y_{n,k})$, we have $\overline{G(x)} = X_{n+1,k}$ and $[x] = Int(X_{n+1,k} \setminus Y_{n,k})$;*
- *for all $x \in Int(Y_{n,k})$, we have $\overline{G(x)} = Y_{n,k}$ and $[x] = Int(Y_{n,k})$;*
- *for all $g \in G$ and for all $i = 1, \ldots, k$, we have $g\,|_{\partial A_i} = \mathbf{1}_{\mathbf{S}^{n-1}}$;*

(3) *for all $g \in G$, g is $\mathbf{1}_\partial$-isotopic to the identity $\mathbf{1}_{\mathbf{D}^{n+1}}$.*

Proof. The proof of this lemma is analogous to those of the preliminary lemmas. In order to satisfy the first item of (2), let us build the three following sets. The first, S^a, mixes the points of the interior of $(\mathbf{S}^n \times [0, \frac{1}{2}])/_{(\mathbf{S}^n \times \{0\}) = \{p\}}$, and is defined by:

$$S^a = \left\{s \in \mathrm{Homeo}(X_{n+1,k}) \text{ such that } s|_{\mathbf{S}^n \times [\frac{1}{2},1]} = \mathbf{1}_{\mathbf{S}^n \times I};\ s|_{(\mathbf{S}^n \times [0,\frac{1}{2}])/_{(\mathbf{S}^n \times \{0\}) = \{p\}}} = g\right\}_{g \in S_7}$$

where S_7 is the finite set of generators of the group built in Lemma 7. The second set, S^b, mixes the points of the interior of $\mathbf{S}^n \times [\frac{1}{2}, 1]$ and is built from the set S_6 generating the group of Lemma 6 by considering:

$$S^b = \left\{s \in \mathrm{Homeo}(X_{n+1,k}) \text{ such that } s|_{(\mathbf{S}^n \times [0,\frac{1}{2}])/_{(\mathbf{S}^n\times\{0\})=\{p\}}} = \mathbf{1}_{\mathbf{D}^{n+1}};\ s|_{(\mathbf{S}^n \times [\frac{1}{2},1])} = g\right\}_{g\in S_6}.$$

The role of the third set S^c is to mix the points of $Int(A_i)$ for all $i = 1, \dots, k$, with the points of $X_{n+1,k} \setminus (\mathbf{S}^n \times \{\frac{1}{2}\})$ by using:

$$S^c = \left\{s \in \mathrm{Homeo}(X_{n+1,k}) \text{ such that } s|_{X_{n+1,k}\setminus(\bigcup_{i=1}^k A_i)} = \mathbf{1}_{X_{n+1,k}\setminus(\bigcup_{i=1}^k A_i)};\ \text{for all } i = 1, \dots, k,\ s|_{A_i\times[\frac{1}{3},\frac{2}{3}]} = g\right\}_{g\in S_7}.$$

Note that the restriction of all $s \in (S^a \cup S^b \cup S^c)$ to $Y_{n,k}$ coincides with the identity map $\mathbf{1}_{Y_{n,k}}$. In order to satisfy the second item of (2), we extend the homeomorphisms of S_8, generating the group of Lemma 8 by using the construction of Lemma 3. Let S^d be the set:

$$S^d = \left\{s \in \mathrm{Homeo}(X_{n+1,k}) \text{ such that } s|_{X_{n+1,k}\setminus(Y_{n,k}\times[\frac{1}{3},\frac{2}{3}])} = \mathbf{1}_{X_{n+1,k}\setminus(Y_{n,k}\times[\frac{1}{3},\frac{2}{3}])};\ s|_{Y_{n,k}\times[\frac{1}{3},\frac{2}{3}]} = \hat{g}\right\}_{g\in S_8}.$$

Let G be the group generated by S^a, S^b, S^c and S^d. All the items hold by construction. □

In order to prove Proposition 13 we have already proved that the proposition is true for rooted trees of height 0 (Lemma 14) and for those of height 1 (Lemma 15). We shall prove, at last, how to proceed by induction in the case of a tree of any height.

Lemma 17. *Let $H \in \mathbf{N}$, $H \geqslant 2$. Assume that all rooted trees of height $h \leqslant H$ are locally realizable in dimension $(h+1)$. Then, all rooted trees of height $(H+1)$ are locally realizable in dimension $(H+2)$.*

Proof. Let Γ be a rooted tree of height $(H+1)$ and denote by v_0 its root. If we remove v_0 (and its outgoing edges) from Γ, we obtain a forest $\mathcal{F}$ whose trees are the maximal connected components of $\Gamma \setminus \{v_0\}$. We can consider $\mathcal{F}$ as the union of the following two disjoint sets:

- $\mathcal{F}_H$ consisting of the trees of $\mathcal{F}$ of height H;
- $\mathcal{F}_{\leqslant(H-1)}$ consisting of the trees of $\mathcal{F}$ of height at most $(H-1)$.

Note that $\mathcal{F}_H$ is necessarily non-empty. We proceed with a three-step construction.

Step 1: local realizability of "$\mathcal{F}_H$ plus one" in dimension $(H+2)$. We apply to each tree R of $\mathcal{F}_H$ the following operations. First, we increase the height of R by 1 by adding a new vertex v and a new oriented edge from v to the root of R, denoted by w. Therefore we have a new tree, denoted by R', of height $(H+1)$ and whose root is the new vertex v. Second, we show that R' is locally realizable in dimension $(H+2)$. Let $z_1, \dots, z_{k(R)}$ be the direct successors of w. We look at $\mathbf{D}^{H+2}$ as $(\mathbf{S}^{H+1} \times I)/_{\mathbf{S}^{H+1}\times\{0\}=\{p\}}$.

On one hand, by applying Lemma 16 to $n = H+1$ and $k = k(R)$ and using the same notations, we realize $[v]$ on $Int(X_{H+2,k(R)} \setminus Y_{H+1,k(R)})$, that is, the complement in $Int(\mathbf{D}^{H+2})$ of the $(H+1)$-dimensional sphere with holes corresponding to $Y_{H+1,k(R)}$, and we realize $[w]$ on the interior of the $(H+1)$-dimensional sphere with holes $Y_{H+1,k(R)}$. Note that all the homeomorphisms appearing up to this stage coincide with the identity $\mathbf{1}_{S^{H+1}}$ on $\partial\mathbf{D}^{H+2}$ and with the identity $\mathbf{1}_{S^H}$ on $\partial Y_{H+1,k(R)}$, that is, on ∂A_i for all $i = 1, \dots, k(R)$.

On the other hand, each z_i is the root of a tree Z_i of height h_i ($h_i \leqslant (H-1)$). Hence, by assumption, it is locally realizable in dimension (h_i+1). By Corollary 12, we can realize each tree Z_i on the sphere $\mathbf{S}^H$ by a finitely generated group G_{z_i} of homeomorphisms of $\mathbf{S}^H$. Using the above notation, we identify the sphere $\mathbf{S}^H$ associated with z_i with the corresponding ∂A_i and we extend each homeomorphism g generating G_{z_i} to $\mathbf{D}^{H+2}$ in the following way. We use Lemma 3 a first time to obtain a homeomorphism $\hat{g}$ defined on a neighborhood C_i of ∂A_i homeomorphic to $\partial A_i \times I$ and contained in the sphere $(\mathbf{S}^{H+1} \times \{\frac{1}{2}\}) \subseteq \mathbf{D}^{H+2}$ containing $Y_{H+1,k(R)}$, in such a way that ∂A_i is identified with $(\partial A_i \times \{\frac{1}{2}\}) \subseteq C_i$. We apply Lemma 3 a second time to extend $\hat{g}$ to a homeomorphism $\hat{\hat{g}}$ defined on a neighborhood D_i of C_i homeomorphic to $\partial C_i \times I$, whose intersection with the sphere $(\mathbf{S}^{H+1} \times \{\frac{1}{2}\}) \subseteq \mathbf{D}^{H+2}$ containing $Y_{H+1,k(R)}$ is C_i, identified with $(C_i \times \{\frac{1}{2}\}) \subseteq D_i$. All the homeomorphisms obtained by this double hat construction can be extended to $\mathbf{D}^{H+2}$ by considering the identity on the complement of their support D_i.

By construction, the local realization of R' in dimension $(H+2)$ is therefore obtained by considering the group $G^R_{loc,H+2}$ generated by the generators of the group of Lemma 16 and the double hat of the generators of the groups G_{z_i}, $i = 1, \dots, k(R)$, realizing each subtree Z_i on $\mathbf{S}^H$, as described above.

Step 2: local realizability of "$\mathcal{F}_{\leqslant(H-1)}$ plus one" in dimension $(H+2)$. We apply to each tree R of $\mathcal{F}_{\leqslant(H-1)}$ the following operations. First, we increase the height of R by 1 by adding a new vertex v and a new oriented edge from v to the root of R, denoted by w. Therefore we have a new tree, denoted by R', having height $h_{R'}$ (at most H) and whose root is the new vertex v. Second, we show that R' is locally realizable in dimension $(H+2)$. By assumption, since $h_{R'} \leqslant H$, R' is locally realizable in dimension $(h_{R'}+1) \leqslant (H+1)$, thus R' is also locally realizable in dimension $(H+2)$ by Lemma 11.

Step 3: fusion of the roots. In the previous steps we have associated with each tree R of the forest $\mathcal{F}$ another tree R' and we have locally realized this new tree in dimension $(H+2)$. The set of the trees R' built this way defines a forest which we call $\mathcal{F}'$. Note that the initial tree Γ can be obtained by the identification of the roots of all tree R' of $\mathcal{F}'$ with v_0, the root of Γ. For this reason, we can locally realize Γ in dimension $(H+2)$ by considering the local realizations of all trees R' and mixing the orbits of the roots.

More precisely, denote by $k(\mathcal{F}')$ the cardinality (= number of trees) of the forest $\mathcal{F}'$, which is also the cardinality of the forest $\mathcal{F}$. Consider, in $Int(\mathbf{D}^{H+2})$, $k(\mathcal{F}')$ $(H+2)$-dimensional disks. Denote them by $B_1, \ldots, B_{k(\mathcal{F}')}$ and assume that they are pairwise disjoint. For each $i = 1, \ldots, k(\mathcal{F}')$ let us identify B_i with the disk locally realizing the i-th tree of $\mathcal{F}'$, denoted by R'_i. Let $S'_{R'_i}$ be the set of the generators of the group associated with the local realization of R'_i. By extending such homeomorphisms to $\mathbf{D}^{H+2}$ we obtain:

$$S_{R'_i} = \left\{ f \in \mathrm{Homeo}\big(\mathbf{D}^{H+2}\big) \text{ such that} f|_{\mathbf{D}^{H+2}\setminus B_i} = \mathbf{1}_{\mathbf{D}^{H+2}\setminus B_i} \text{ and } f|_{B_i} = g \right\}_{g \in S'_{R'_i}}.$$

Next, for each B_i, $i = 1, \ldots, k(\mathcal{F}')$, denote by A_i the disk corresponding to $\overline{B_{loc,H+2}}$, see Definition 9. Following Lemma 8, consider the space $Y_{H+2,k(\mathcal{F}')+1} = \mathbf{D}^{H+2} \setminus (\bigcup_{i=1}^{k(\mathcal{F}')} Int(A_i))$ and let S_Y be the set of the generators of the group built in Lemma 8. By extending the homeomorphisms contained in S_Y to $\mathbf{D}^{H+2}$ we obtain:

$$\tilde{S}_Y = \left\{ f \in \mathrm{Homeo}\big(\mathbf{D}^{H+2}\big) \text{ such that } f|_{A_i} = \mathbf{1}_{A_i} \text{ and } f|_{Y_{H+2,k(\mathcal{F}')+1}} = g \right\}_{g \in S_Y}.$$

The group G generated by:

$$\left(\bigcup_{i=1}^{k(\mathcal{F}')} S_{R'_i} \right) \cup \tilde{S}_Y$$

is therefore finitely generated by construction and locally realizes the initial tree Γ in dimension $(H+2)$. $\square$

5. Realization of graphs by compact spaces

In this section we prove that any finite Hasse diagram of height n can be realized on a compact CW-complex of dimension $(n+1)$.

Proposition 18. *Let Γ be a finite Hasse diagram. Then there exist a compact topological space X and a finitely generated group $G \subseteq \mathrm{Homeo}(X)$ such that (X, G) realizes Γ.*

The proof consists of two parts. First, we "open up" the original graph Γ in order to obtain a forest $\mathcal{F}$. This procedure is reversible: by identifying some vertices and edges of $\mathcal{F}$ we can "close $\mathcal{F}$ up" and get Γ back. This procedure will be defined in Section 5.1. Second, in Section 5.2, we show how to realize each tree of $\mathcal{F}$ in a way that is compatible with the above closing up, thus we obtain a realization of Γ by identification.

5.1. Decomposition of a graph into a maximal forest

Given an oriented graph Γ, for each vertex v one can define the *graph growing from v*, that is, the graph whose vertices are all the vertices of Γ which can be reached by an oriented path starting from v. The purpose of this definition is to help us to describe a recursive procedure associating with any given Hasse diagram Γ, a canonical forest $\mathcal{F}$ with the property that the graph Γ is obtained from $\mathcal{F}$ by identification of some of its vertices and edges.

Step 0: choice of the roots of the trees of $\mathcal{F}$. Let *Roots* be the set of the vertices of Γ with no incoming edges. For any $v \in Roots$ we shall build a rooted tree T_v with root v, by applying the following recursive procedure to the graph growing from v, denoted by $\Delta(v)$ and computed from Γ.

Step 1: trivial case. If $\Delta(v)$ has v as a unique vertex and no edges, the result T_v of the procedure is $\Delta(v)$ itself.

Step 2: recursive step. If $\Delta(v)$ has height at least 1, then the vertex v in T_v has as many outcoming edges as v has in Γ. In T_v, as well as in Γ, the endpoints of such edges are all distinct. Denote them by $w_1, \ldots, w_{k(v)}$ ($w_i \in T_v$, $w_i \neq w_j$ if $i \neq j$), and denote by $\varphi(w_i)$ the vertex in Γ corresponding to $w_i \in T_v$. Then each w_i is the root of a subtree $T_{w_i} \subseteq T_v$ obtained by applying the recursive procedure to $\Delta(w_i)$ in such a way that T_{w_i} and T_{w_j} are disjoint if $i \neq j$.

Let $(V_{\mathcal{F}}, E_{\mathcal{F}})$ and (V_Γ, E_Γ) be the sets of vertices and edges of the forest $\mathcal{F}$ and of the graph Γ respectively. Note that by construction the forest $\mathcal{F}$ is provided with a map $\varphi : V_{\mathcal{F}} \to V_\Gamma$ such that two vertices have the same image if and only if they are copies of the same vertex of Γ. The map φ induces a map between the set of edges $E_{\mathcal{F}}$ and E_Γ: the image of the unique edge connecting z to w in $\mathcal{F}$ is the unique edge connecting $\varphi(z)$ to $\varphi(w)$ in Γ. We have

$$\Gamma = \mathcal{F}/_{\varphi}.$$

Note that if two vertices u and v of $V_{\mathcal{F}}$ are mapped to the same vertex of V_{Γ}, then the trees growing from u and v and computed from $\mathcal{F}$ are isomorphic and they are mapped by φ to the graph growing from $\varphi(u) = \varphi(v)$ and computed from Γ.

5.2. General construction

Let Γ be a Hasse diagram and $\mathcal{F}$ be the forest obtained from the procedure described in the previous subsection. We realize simultaneously all the trees of $\mathcal{F}$.

Initial steps. We first consider the vertices z belonging to a tree of $\mathcal{F}$ and which are the copy of a vertex of Γ lying in the ground floor of Γ. We realize the corresponding classes $[z]$ on a one-dimensional sphere $\mathbf{S}^1$, endowed with an irrational rotation (Lemma 5). In particular, if two vertices z_1 and z_2 are copies of the same vertex of Γ, we take care of considering the same dynamics on them. We extend the construction to the vertices w belonging to a tree of $\mathcal{F}$ and which are the copy of a vertex of Γ lying in the first floor of Γ. By using Lemma 8, we realize the corresponding classes $[w]$ on the interior of a two-dimensional sphere with holes $Y_{2,k(w)}$, where $k(w)$ is the number of outgoing edges of w. Note that if w_1 and w_2 are copies of the same vertex in Γ, that is, if $\varphi(w_1) = \varphi(w_2)$, then $k(w_1) = k(w_2)$ and we take care of considering the same dynamics on the associated sphere with holes $Y_{2,k(w_i)}$. We take the same care when extending the rotations defined on the 1-spheres $\mathbf{S}^1$ which are the boundary of $Y_{2,k(w_i)}$ and which realize the ground floor vertices.

General steps. By combining the results and constructions of the previous sections, we realize the vertices z for which $\varphi(z)$ has even height $2j$ on the interior of a $(2j+1)$-dimensional sphere from which we have removed the space realizing the vertices t such that $\varphi(t)$ has height strictly less than $2j$. Note that if two vertices z_1 and z_2 are copies of the same vertex of Γ, we take care of considering the same dynamics on them. In the same way, we realize the vertices w for which $\varphi(w)$ has odd height $2j+1$ on the interior of a $(2j+2)$-dimensional sphere with holes $Y_{2j+2,k(w)}$, where $k(w)$ is the number of outgoing edges of w. Again, if two vertices are copies of the same vertex of Γ, we take care of considering the same dynamics on them.

Final step. Note that according to the above construction, each tree of $\mathcal{F}$ with root v is realized either on a sphere, if the height of $\varphi(v)$ is even, or on a sphere with holes $Y_{n,k}$ for some n and k, if the height of $\varphi(v)$ is odd. Let $X_{\mathcal{F}}$ be the (finite disjoint) union of these compact spaces and let us identify the class $[x]$ with the associated vertex, so that the map φ is now defined on the classes. Let $G_{\mathcal{F}}$ be the finitely generated group acting on $X_{\mathcal{F}}$ built in the previous steps in such a way that if for $x \in X_{\mathcal{F}}$ and $g \in G_{\mathcal{F}}$ we have that if $supp(g) \cap [x] \neq \emptyset$, then $supp(g) \cap [y] \neq \emptyset$ for all $[y] \in \varphi^{-1}(\varphi([x]))$. Moreover, the dynamics of g is the same on each $[y] \in \varphi^{-1}(\varphi([x]))$. Recall that $\Gamma = \mathcal{F}/_{\varphi}$. Because of the care which we took in the previous steps of the construction, we can not only identify the classes of $X_{\mathcal{F}}$ associated via φ with the same class of Γ, but we can also identify the dynamics acting on them. In other words, we are saying that $(X_{\mathcal{F}}/_{\varphi}, G_{\mathcal{F}}/_{\varphi})$ realizes Γ. We can also observe that $X_{\mathcal{F}}/_{\varphi}$ is compact because the quotient under φ is made on compact spaces (that is, the spheres or the spheres with holes associated with the trees growing from the classes having the same image under φ). Moreover, $X_{\mathcal{F}}/_{\varphi}$ is a CW-complex by construction.

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References

[1] C. Bonatti, H. Hattab, E. Salhi, Quasi-orbits spaces associated to T_0-spaces, Fund. Math. 211 (2011) 267–291.
[2] E. Bouacida, O. Echi, E. Salhi, Feuilletage et topologie spectrale, J. Math. Soc. Japan 52 (2) (2000) 447–464.
[3] N. Bourbaki, Topologie générale, chapitre 1 à 4, Masson, 1990.
[4] A. Grothendieck, J. Dieudonné, Éléments de Géométrie Algébrique, Die Grundlehren der Mathematischen Wissenschaften, vol. 166, Springer-Verlag, New York, 1971.
[5] E.G. Effros, F. Hahn, Locally compact transformation groups and C^*-algebras, Mem. Amer. Math. Soc. 75 (1967).
[6] C. Godbillon, Dynamical Systems on Surfaces, Universitext, Springer-Verlag, Berlin, New York, 1983.
[7] C. Godbillon, Feuilletages. Etudes géométriques, Birkhäuser-Verlag, 1991.
[8] H. Hattab, E. Salhi, Groups of homeomorphisms and spectral topology, Topology Proc. 28 (2) (2004) 503–526.
[9] E. Salhi, Niveau des feuilles, C. R. Acad. Sci. Paris 301 (1985) 219–222.